

Decomposition results for stochastic storage processes and queues with alternating Lévy inputs

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Abstract

In this paper we generalize known workload decomposition results for Lévy queues with secondary jump inputs and queues with server vacations or service interruptions. Special cases are polling systems with either compound Poisson or more general Lévy inputs. Our main tools are new martingale results, which have been derived in a companion paper.

Keywords: Lévy-type processes, Lévy storage systems, Kella-Whitt martingale, decomposition results, queues with server vacations

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1 Introduction

It is well known in queueing theory (e.g., [11, 18]) that in a stable M/G/1 queue with server down periods (vacations, interruptions, etc.) the steady state waiting time distribution (properly defined) is a convolution of two or more distributions, one of which is always the steady state waiting time distribution of an ordinary M/G/1 queue. As Poisson arrivals see time averages, this result also holds for the workload

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process. [16] studies a more general model of a Lévy process with no negative jumps and additional jumps that occur at stopping epochs and the size of which is measurable with respect to the current information. The interesting outcome of [16] was that the same (and even more general) decomposition results that were known for queues also turned out to hold for these Lévy processes with additional jumps.

That model is interesting in its own right but can also be viewed as a weak limit of queues with down (off) periods where during these down periods workload can only accumulate as the server is idle. Consider a process that can be either in an *up* (on) state or a *down* state. When it is in the *up* state it behaves like some Lévy process with no negative jumps and a negative drift. When it is in a *down* state it behaves like a subordinator, that is, a nondecreasing Lévy process. The question that comes to mind is whether this up/down process (for which we give a precise definition later) obeys a similar decomposition property. This would immediately imply a decomposition in certain polling systems as described in Section 5 below. It is a simple observation that if one cuts and pastes the up/down process such that only the up periods are visible, then the resulting process is the one that was considered in [16]. As it seemed that the results of [16] could not be used in our setting, we found it necessary to develop a more general theory, in particular a certain martingale theory that would streamline our work and could be useful in other applications as well. That direction was developed in [15].

The first main result of [15] is the extension of the martingale results of [17] to the case where the driving process is a Lévy-type process. That is, it is a sum of stochastic integrals of some bounded left continuous right limit process with respect to coordinate processes associated with some multidimensional Lévy process. Such processes with an even more general (predictable) integrand are discussed in [1]. The second main result of [15] is that our local martingale is in fact an L^2 martingale, and moreover, when upon dividing by the time parameter t it converges to zero almost surely and in L^2 as $t \rightarrow \infty$.

The main goal of the present paper is to apply the martingale results which were derived in [15] to establish decomposition results for the up/down model that was introduced above.

The paper is organized as follows. In Theorem 1 of Section 2 we summarize the main results from [15] which are needed in the present paper. In Section 3 we apply our results to establish decomposition results for the up/down model, thereby considerably generalizing the results of [16]. In Section 4 we identify the non-standard component in the decomposition associated with down periods. Finally in Section 5 a discussion of polling systems, the motivation for this study, is given and the contribution of our results to this area is emphasized.

For extensive discussions of decomposition results in queues and

storage processes, we refer to the surveys [3, 9] and to the recent study [12] and references therein.

2 Preliminaries

In preparation of our analysis and in order to make this paper more self contained we first recall the main results from [15] which we will need here.

For what follows, given a càdlàg (right continuous left limit) function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$, we denote $g(t-) = \lim_{s \uparrow t} g(s)$, $\Delta g(t) = g(t) - g(t-)$ with the convention that $\Delta g(0) = g(0)$ and if g is VF (finite variation on finite intervals), then $g^d(t) = \sum_{0 \leq s \leq t} \Delta g(s)$ and $g^c(t) = g(t) - g^d(t)$. Also, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$ and *a.s.* abbreviates *almost surely*. Finally $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, $a^- = a \wedge 0$ and $a^+ = a \vee 0$.

Let $X = (X_1, \dots, X_K)$ be a càdlàg K -dimensional Lévy process with respect to some standard filtration $\{\mathcal{F}_t | t \geq 0\}$ having no negative jumps (the Lévy measure is concentrated on \mathbb{R}_+^K) with Laplace-Stieltjes exponent

$$\begin{aligned} \psi(\gamma) &= \log E e^{-\gamma^T X(1)} = -c^T \gamma + \frac{\gamma^T \Sigma \gamma}{2} \\ &\quad + \int_{\mathbb{R}_+^K} \left(e^{-\gamma^T x} - 1 + \gamma^T x 1_{\{\|x\| \leq 1\}} \right) \nu(dx), \end{aligned} \quad (1)$$

where $\gamma \in \mathbb{R}_+^K$. It is well known that in this case $\psi(\gamma)$ is finite for each $\gamma \geq 0$, convex (thus continuous) with $\psi(0) = 0$, infinitely differentiable in the interior of \mathbb{R}_+^K , and for every $\gamma \geq 0$ for which $\gamma^T X$ is not a subordinator (not nondecreasing), $\psi(t\gamma) \rightarrow \infty$ as $t \rightarrow \infty$. Furthermore, $EX_k(t) = -t \frac{\partial \psi}{\partial \gamma_k}(0+)$ (finite or $+\infty$, but can never be $-\infty$) and when the first two right derivatives at zero are finite, then $\text{Cov}(X_k(t), X_\ell(t)) = t \frac{\partial^2 \psi}{\partial \gamma_k \partial \gamma_\ell}(0+)$.

Let $I = (I_1, \dots, I_K)$ be a nonnegative, bounded, càdlàg and adapted process and define (a special case of) a Lévy-type process as a sum of the following stochastic integrals.

$$\tilde{X}(t) = \sum_{k=1}^K \int_{[0,t]} I_k(s-) dX_k(s). \quad (2)$$

Finally, let Y be a càdlàg adapted process having a.s. finite variation on finite intervals, set

$$Z(t) = \tilde{X}(t) + Y(t), \quad (3)$$

and assume that Z is bounded below. Under the above setup, the following summarizes what we need from [15].

Theorem 1 *Given the assumption above:*

(i) *The following is a mean square martingale having zero mean:*

$$\begin{aligned} M(t) &= \int_0^t \psi(I(s))e^{-Z(s)}ds + e^{-Z(0)} - e^{-Z(t)} - \int_0^t e^{-Z(s)}dY^c(s) \\ &\quad + \sum_{0 < s \leq t} e^{-Z(s)} \left(1 - e^{\Delta Y(s)}\right). \end{aligned} \quad (4)$$

(ii) $M(t)/t \rightarrow 0$ as $t \rightarrow \infty$ a.s. and in L^2 .

(iii) *If*

$$\frac{1}{t} \int_0^t I_k(s)ds \rightarrow \beta_k, \quad (5)$$

a.s. as $t \rightarrow \infty$ for each k , and if $EX_k(1) < \infty$ ($EX_k(1)^- > -\infty$ as there are no negative jumps), then

$$\frac{\tilde{X}(t)}{t} \rightarrow \sum_{k=1}^K \beta_k EX_k(1) \quad (6)$$

a.s. as $t \rightarrow \infty$.

(iv) *When $Y(t) = -\inf_{0 \leq s \leq t} \tilde{X}(s)^-$ and (6) holds, then*

$$\frac{1}{t} \int_0^t \psi(I(s))e^{-Z(s)}ds \rightarrow -\left(\sum_{k=1}^K \beta_k EX_k(1)\right)^-, \quad (7)$$

a.s. as $t \rightarrow \infty$.

3 Decomposition results for Lévy storage processes

In this section we complement the results of [16] as follows. Let $0 = T_0 \leq S_1 \leq T_1 \leq S_2 \leq T_2 \dots$ be an increasing sequence of a.s. finite stopping times with respect to the standard filtration $\{\mathcal{F}_t | t \geq 0\}$ satisfying $T_{n-1} < T_n$ and $T_n \rightarrow \infty$ a.s. Let $X_n = S_n - T_{n-1}$ and $Y_n = T_n - S_n$. The model here is that $(T_{n-1}, S_n]$ with lengths X_n are *down* periods, where there is no output (the “server” is not working) and therefore the buffer content can only accumulate. $(S_n, T_n]$ with length Y_n are *up* periods where there is both input and output, which is modeled as usual by a reflected (Skorohod map of the) process.

Remark 1 We note that in some models it is possible that there is no reflection. This occurs, for example, whenever the server is shut off as soon as the system empties, which may be modeled via the stopping times.

Let X_u be a one-dimensional càdlàg Lévy process with no negative jumps which is not a subordinator (not nondecreasing), and with Laplace-Stieltjes exponent

$$\varphi(\alpha) = -c_u\alpha + \frac{\sigma_u^2\alpha^2}{2} + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{\{x \leq 1\}}) \nu_u(dx) \quad (8)$$

and assume that $EX_u(1) = -\varphi'(0) = c_u + \int_{(1,\infty)} x\nu_u(dx) < 0$. This models the net input process (input minus potential output) during up periods. Let X_d be a one-dimensional right continuous subordinator (nondecreasing Lévy process) with Laplace-Stieltjes exponent $-\eta$ where

$$\eta(\alpha) = c_d\alpha + \int_{(0,\infty]} (1 - e^{-\alpha x}) \nu_d(dx) \quad (9)$$

and assume that $EX_d(1) = \eta'(0) < \infty$. The latter models the process according to which work accumulates during down periods.

Now, set $N(t) = \sup\{n \mid T_n \leq t\}$ and let $J(t) = 1_{\{S_{N(t)+1} > t\}}$ and thus $J(t) = 1_{\{J(t)=1\}}$ and $1 - J(t) = 1_{\{J(t)=0\}}$. That is, $J(t) = 1$ during down periods and $J(t) = 0$ during up periods. Finally, for $W(0) \in \mathcal{F}_0$ let

$$\begin{aligned} \tilde{X}_d(t) &= \int_{(0,t]} J(s-) dX_d(s), \\ \tilde{X}_u(t) &= \int_{(0,t]} (1 - J(s-)) dX_u(s), \\ \tilde{X}(t) &= \tilde{X}_u(t) + \tilde{X}_d(t), \\ L(t) &= - \inf_{0 \leq s \leq t} (W(0) + \tilde{X}(s))^- , \\ W(t) &= W(0) + \tilde{X}(t) + L(t) . \end{aligned} \quad (10)$$

The process $\{W(t) \mid t \geq 0\}$ is the content process of interest for which we would like to establish a general decomposition. During down periods it behaves like a subordinator with exponent $-\eta$ (and only grows) and during up periods it behaves like Lévy process with exponent φ and is reflected at the origin. This general decomposition is given by the following theorem which will be interpreted after its proof.

Theorem 2 *If, in addition to the above setup and assumptions,*

$$\frac{1}{t} \int_0^t e^{-\alpha W(s)} ds \rightarrow Ee^{-\alpha W(\infty)} \quad (11)$$

a.s. as $t \rightarrow \infty$ (ergodic convergence) for some finite random variable $W(\infty)$ and

$$\frac{1}{t} \int_0^t J(s) ds \rightarrow p_d \leq \frac{\varphi'(0)}{\eta'(0) + \varphi'(0)}, \quad (12)$$

then there exists a nonnegative random variable W_d such that if $p_d > 0$ then a.s.

$$\frac{\int_0^t e^{-\alpha W(s)} J(s) ds}{\int_0^t J(s) ds} \rightarrow E e^{-\alpha W_d} \quad (13)$$

for every $\alpha \geq 0$. Moreover, with

$$\pi_\ell = 1 - \left(1 + \frac{\eta'(0)}{\varphi'(0)}\right) p_d \quad (14)$$

and

$$\pi = \frac{\eta'(0)}{\eta'(0) + \varphi'(0)} \quad (15)$$

we have that

$$\begin{aligned} E e^{-\alpha W(\infty)} &= \pi_\ell \frac{\varphi'(0)\alpha}{\varphi(\alpha)} \\ &+ (1 - \pi_\ell) \left(1 - \pi + \pi \frac{\eta(\alpha)}{\eta'(0)\alpha} \frac{\varphi'(0)\alpha}{\varphi(\alpha)}\right) E e^{-\alpha W_d}. \end{aligned} \quad (16)$$

Proof: With $\psi(\gamma_1, \gamma_2) = \varphi(\gamma_1) - \eta(\gamma_2)$ (or any other ψ with $\psi(\alpha, 0) = \varphi(\alpha)$ and $\psi(0, \alpha) = -\eta(\alpha)$), $I_1(s) = \alpha(1 - J(s))$ and $I_2(s) = \alpha J(s)$, Theorem 1-(iv) and (12) imply that

$$\begin{aligned} &\frac{1}{t} \int_0^t (\varphi(\alpha)(1 - J(s)) - \eta(\alpha)J(s)) e^{-\alpha W(s)} ds \\ &= \varphi(\alpha) \frac{1}{t} \int_0^t e^{-\alpha W(s)} ds - (\varphi(\alpha) + \eta(\alpha)) \frac{1}{t} \int_0^t J(s) e^{-\alpha W(s)} ds \end{aligned} \quad (17)$$

converges a.s., as $t \rightarrow \infty$, to

$$-\alpha(-(1 - p_d)\varphi'(0) + p_d\eta'(0))^- = \alpha((1 - p_d)\varphi'(0) - p_d\eta'(0)) \quad (18)$$

the last equality being due to $p_d \leq \frac{\varphi'(0)}{\eta'(0) + \varphi'(0)}$.

Now, by the convergence of (17) and by (11) we have that

$$\frac{1}{t} \int_0^t J(s) e^{-\alpha W(s)} ds \quad (19)$$

converges almost surely to some limit and by (12) so does

$$\frac{\int_0^t J(s) e^{-\alpha W(s)} ds}{\int_0^t J(s) ds} = \frac{\frac{1}{t} \int_0^t J(s) e^{-\alpha W(s)} ds}{\frac{1}{t} \int_0^t J(s) ds} \quad (20)$$

Next, observe that for each $t \geq 0$ (and each ω in the sample space) for which $\int_0^t J(s)ds > 0$ we have that the expression on the right hand side of (20) is the Laplace-Stieltjes transform of an a.s. nonnegative and finite random variable and thus if this ratio converges to some constant $g(\alpha)$ for each α then g must be a Laplace-Stieltjes transform of some nonnegative (not necessarily a.s. finite) random variable which we denote by W_d . If in addition $g(\alpha) \rightarrow 1$ as $\alpha \downarrow 0$ then necessarily g is the Laplace-Stieltjes transform of a proper distribution on \mathbb{R}_+ and this is the case at hand as can be seen from (but is not needed for) the end result (16).

Finally, note that by (17), (18), (20) and the above discussion we have that

$$\begin{aligned} \varphi(\alpha)Ee^{-\alpha W(\infty)} - (\varphi(\alpha) + \eta(\alpha))p_d Ee^{-\alpha W_d} \\ = \alpha((1 - p_d)\varphi'(0) - p_d\eta'(0)) \end{aligned} \quad (21)$$

which is equivalent to (16) via some obvious manipulations. \blacksquare

Let us now interpret (16). First we note that, since $\varphi'(0) > 0$ then $\frac{\alpha\varphi'(0)}{\varphi(\alpha)}$ is the Laplace-Stieltjes transform of the stationary, limit and ergodic distribution associated with the process $Z_u(t) = X_u(t) + L_u(t)$ where $L_u(t) = -\inf_{0 \leq s \leq t} X_u(s)$, as well as the Laplace-Stieltjes transform of the random variable $\sup_{s \geq 0} X_u(s)$. This is well known and there are quite a few proofs of this generalized Pollaczek-Khinchin formula in the literature, one of which is in [17].

Next we observe that from [14], $\frac{\eta(\alpha)}{\alpha\eta'(0)}$ is the Laplace-Stieltjes transform of the stationary excess lifetime distribution associated with the jumps of the subordinator X_d . For ease of reference simply observe that from

$$\eta(\alpha) - c_d\alpha = \int_{(0,\infty)} (1 - e^{-\alpha x})\nu_d(dx) = \alpha \int_0^\infty e^{-\alpha x}\nu(x, \infty)dx \quad (22)$$

and $\eta'(0) = c_d + \bar{\nu}_d$, where $\bar{\nu}_d = \int_{(0,\infty)} x\nu_d(dx) = \int_0^\infty \nu(x, \infty)dx$, we have that

$$\frac{\eta(\alpha)}{\alpha\eta'(0)} = \frac{c_d}{c_d + \bar{\nu}_d} + \frac{\bar{\nu}_d}{c_d + \bar{\nu}_d} \int_0^\infty e^{-\alpha x} \frac{\nu(x, \infty)}{\bar{\nu}_d} dx \quad (23)$$

which is the Laplace-Stieltjes transform of the following distribution function:

$$F_e(y) = \frac{c_d}{c_d + \bar{\nu}_d} + \frac{\bar{\nu}_d}{c_d + \bar{\nu}_d} \int_0^y \frac{\nu(x, \infty)}{\bar{\nu}_d} dx \quad (24)$$

for $y \geq 0$ and $F_e(y) = 0$ for $y < 0$. This is a somewhat generalized stationary excess lifetime distribution associated with the jumps of X_d .

Now introduce the random variables W_u, Y_e, I_l, I :

- $W_u \sim \sup\{X_u(s) \mid s \geq 0\}$ with $Ee^{-\alpha W_u} = \frac{\alpha\varphi'(0)}{\varphi(\alpha)}$,
- $Y_e \sim F_e$ with $Ee^{-\alpha Y_e} = \frac{\eta(\alpha)}{\eta'(0)\alpha}$,
- $P(I_\ell = 1) = 1 - P(I_\ell = 0) = \pi_\ell$,
- $P(I = 1) = 1 - P(I = 0) = \pi$,
- W_d and $W(\infty)$ are as in Theorem 2.

Then

Theorem 3 *Under the conditions of Theorem 2, (16) is equivalent to*

$$W(\infty) \sim I_\ell W_u + (1 - I_\ell)(I(W_u + Y_e) + W_d), \quad (25)$$

where W_u, Y_e, I_ℓ, I, W_d are assumed independent.

We note that replacing the two instances of W_u in (25) by two different i.i.d. random variables distributed like W_u would not change the overall distribution.

One important special case of this model is when during up periods, whenever there is a positive content, the input has the same law as during down periods and the output is at a fixed rate $r > 0$. That is, $\varphi(\alpha) = \alpha r - \eta(\alpha)$. A special case of this model was studied in [14]. In this particular case it is easy to check that (as in Equation (4.12) of [14])

$$1 - \pi + \pi \frac{\eta(\alpha)}{\eta'(0)\alpha} \frac{\varphi'(0)\alpha}{\varphi(\alpha)} = \frac{\alpha\varphi'(0)}{\varphi(\alpha)}, \quad (26)$$

that is, that $I(W_u + Y_e) \sim W_u$. So in this case we have the following.

Corollary 1 *When $\varphi(\alpha) = \alpha r - \eta(\alpha)$ then*

$$W(\infty) \sim W_u + (1 - I_\ell)W_d, \quad (27)$$

where W_u, I_ℓ and W_d are independent; and when in addition $\ell = 0$ (equivalently $\tilde{X}(t)/t \rightarrow 0$ or $p_d = 1 - \pi = \frac{\varphi'(0)}{\eta'(0) + \varphi'(0)} = 1 - \frac{\eta'(0)}{r}$), then

$$W(\infty) \sim W_u + W_d, \quad (28)$$

where W_u and W_d are independent.

We note that in Corollary 1 the term $\pi = \frac{\eta'(0)}{r}$ may be referred to as the *traffic intensity* and is consistent with queueing theory.

Remark 2 Throughout this and the following section we are focussing on almost sure convergence. However, throughout, most “almost sure” statements could be trivially replaced by “in probability” without changing anything else (simply by looking at subsequences that converge a.s.). We are not aware of related applications where the convergence is in probability but not almost surely and thus did not see a point in making this issue more precise.

Remark 3 In [16] the focus is on convergence in distribution rather than long run a.s. convergence. As in the previous remark, we could follow the same ideas with similar proofs (but with more restrictive assumptions). We chose to leave this out as, given what follows, and what is already available in [16], it may be considered an exercise.

4 How to interpret W_d ?

In this section we identify the non-standard component in the decomposition of Theorem 2, associated with down periods. In particular, we will express the Laplace-Stieltjes transform $\frac{\eta(\alpha)}{\alpha\eta'(0)}Ee^{-\alpha W_d}$ of $Y_e + W_d$ in terms of the transforms of the workloads at the ends of up and down periods.

We recall that under the assumptions of Theorem 2,

$$Ee^{-\alpha W_d} = \lim_{t \rightarrow \infty} \frac{\int_0^t e^{-\alpha W(s)} J(s) ds}{\int_0^t J(s) ds} \quad (29)$$

and since for every nonnegative random variable V we have that $e^{-\alpha V} = \alpha \int_0^\infty e^{-\alpha x} 1_{\{V \leq x\}} dx$, then also here

$$\frac{\int_0^t e^{-\alpha W(s)} J(s) ds}{\int_0^t J(s) ds} = \alpha \int_0^\infty e^{-\alpha x} \frac{\int_0^t 1_{\{W(s) \leq x\}} J(s) ds}{\int_0^t J(s) ds} dx \quad (30)$$

and thus, a.s., $\frac{\int_0^t 1_{\{W(s) \in \cdot\}} J(s) ds}{\int_0^t J(s) ds}$ (probability distribution valued process) converges in distribution to W_d . This holds in particular if we replace t by S_n . In this case $\int_0^{S_n} J(s) ds = \sum_{k=1}^n X_k$ and thus we have that

$$\frac{\int_0^{S_n} 1_{\{W(s) \in \cdot\}} J(s) ds}{\int_0^{S_n} J(s) ds} = \frac{\sum_{k=1}^n \int_0^{X_k} 1_{\{W(T_{k-1}+s) \in \cdot\}} ds}{\sum_{k=1}^n X_k} \quad (31)$$

where for $s \in [0, X_n]$ we have that

$$W(T_{n-1} + s) = W(T_{n-1}) + X_d(T_{n-1} + s) - X_d(T_{n-1}) \quad (32)$$

and thus

$$\int_0^{X_n} e^{-\alpha W(T_{n-1}+s)} ds = e^{-\alpha W(T_{n-1})} \int_0^{X_n} e^{-\alpha (X_d(T_{n-1}+s) - X_d(T_{n-1}))} ds. \quad (33)$$

Now, since T_{n-1}, S_n are stopping times with respect to $\{\mathcal{F}_t \mid t \geq 0\}$, X_n is a stopping time with respect to $\{\mathcal{F}_{T_{n-1}+t} \mid t \geq 0\}$ (of course, not with respect to the original filtration in general). Moreover, $W(T_{n-1}) \in \mathcal{F}_{T_{n-1}}$ and by the strong Markov property $X_d^{T_{n-1}} \equiv \{X_d(T_{n-1} + t) -$

$X_d(T_{n-1})| t \geq 0\}$ is a subordinator with respect to $\{\mathcal{F}_{T_{n-1}+t} | t \geq 0\}$ with exponent η (that is, distributed like X_d) and is independent of $F_{T_{n-1}}$ (thus, of $W(T_{n-1})$). Thus from [17] we have that

$$-\eta(\alpha) \int_0^t e^{-\alpha X_d^{T_{n-1}}(s)} ds + 1 - e^{-\alpha X_d^{T_{n-1}}(t)} \quad (34)$$

is a zero mean martingale with respect to $\{\mathcal{F}_{T_{n-1}+t} | t \geq 0\}$ and thus by the optional stopping theorem together with monotone and bounded convergence where appropriate we have with

$$\Delta_n = -\eta(\alpha) \int_0^{X_n} e^{-\alpha X_d^{T_{n-1}}(s)} ds + 1 - e^{-\alpha X_d^{T_{n-1}}(X_n)}, \quad (35)$$

that $E[\Delta_n | \mathcal{F}_{T_{n-1}}] = 0$. Moreover, from Lemma 3 of [15] and the fact that $M(t)^2 - [M, M](t)$ is a (zero mean) martingale, we can conclude that when X_n is a.s. finite then

$$E[\Delta_n^2 | \mathcal{F}_{T_{n-1}}] = (2\eta(\alpha) - \eta(2\alpha)) E \left[\int_0^{X_n} e^{-2\alpha X_d^{T_{n-1}}(s)} ds \middle| \mathcal{F}_{T_{n-1}} \right] \quad (36)$$

and in the same way that led to $E[\Delta_n | \mathcal{F}_{T_{n-1}}] = 0$, by substituting 2α instead of α , we have that

$$\eta(2\alpha) E \left[\int_0^{X_n} e^{-2\alpha X_d^{T_{n-1}}(s)} ds \middle| \mathcal{F}_{T_{n-1}} \right] = 1 - E \left[e^{-2\alpha X_d^{T_{n-1}}(X_n)} \middle| \mathcal{F}_{T_{n-1}} \right] \quad (37)$$

and we conclude that

$$E[\Delta_n^2 | \mathcal{F}_{T_{n-1}}] = \left(\frac{2\eta(\alpha)}{\eta(2\alpha)} - 1 \right) \left(1 - E \left[e^{-2\alpha X_d^{T_{n-1}}(X_n)} \middle| \mathcal{F}_{T_{n-1}} \right] \right). \quad (38)$$

In particular, upon multiplying by $e^{-\alpha W(T_{n-1})} \in \mathcal{F}_{T_{n-1}}$, we have that

$$\sum_{k=1}^n e^{-\alpha W(T_{k-1})} \Delta_k \quad (39)$$

is a zero mean martingale, where

$$E \left[\left(e^{-\alpha W(T_{k-1})} \Delta_k \right)^2 \middle| \mathcal{F}_{T_{k-1}} \right] \leq \frac{2\eta(\alpha)}{\eta(2\alpha)} - 1 < \infty. \quad (40)$$

It is well known (cf. Theorem 3 on p. 243 of [10]) that an L^2 martingale M_n satisfying

$$\sum_{k=1}^{\infty} \frac{E(M_k - M_{k-1})^2}{k^2} < \infty \quad (41)$$

also satisfies $M_n/n \rightarrow 0$ a.s. and in L^2 and thus

$$\frac{1}{n} \sum_{k=1}^n e^{-\alpha W(T_{k-1})} \Delta_k \rightarrow 0 \quad (42)$$

a.s. and in L^2 and we finally have the following.

Theorem 4 *Under the assumptions of Theorem 2,*

$$\frac{1}{n} \left(-\eta(\alpha) \int_0^{S_n} e^{-\alpha W(s)} J(s) ds + \sum_{k=1}^n \left(e^{-\alpha W(T_{k-1})} - e^{-\alpha W(S_k)} \right) \right) \rightarrow 0 \quad (43)$$

a.s. and in L^2 , and if in addition $p_d > 0$ then

$$\frac{-\eta(\alpha) \int_0^{S_n} e^{-\alpha W(s)} J(s) ds + \sum_{k=1}^n \left(e^{-\alpha W(T_{k-1})} - e^{-\alpha W(S_k)} \right)}{\int_0^{S_n} J(s) ds} \rightarrow 0 \quad (44)$$

and thus

$$\frac{\sum_{k=1}^n \left(e^{-\alpha W(T_{k-1})} - e^{-\alpha W(S_k)} \right)}{\sum_{k=1}^n X_k} \rightarrow \eta(\alpha) E e^{-\alpha W_d}. \quad (45)$$

Now, note that from $\frac{1}{t} \int_0^t J(s) ds \rightarrow p_d > 0$, if also $T_n/n \rightarrow \mu > 0$ a.s. (and thus also $S_n/n \rightarrow \mu$) then

$$\frac{1}{n} \sum_{k=1}^n X_k = \frac{S_n}{n} \frac{1}{S_n} \int_0^{S_n} 1_{\{J(s)\}} ds \rightarrow \mu p_d > 0 \quad (46)$$

and thus

$$\frac{1}{n} \sum_{k=1}^n \left(e^{-\alpha W(T_{k-1})} - e^{-\alpha W(S_k)} \right) \quad (47)$$

converges a.s. In particular, we have:

Theorem 5 *Under the assumptions of Theorem 2, if $p_d > 0$ and $T_n/n \rightarrow \mu > 0$ a.s., then $\frac{1}{n} \sum_{k=1}^n e^{-\alpha W(S_k)} \rightarrow E e^{-\alpha W_+}$ a.s. for some nonnegative random variable W_+ if and only if $\frac{1}{n} \sum_{k=1}^n e^{-\alpha W(T_{k-1})} \rightarrow E e^{-\alpha W_-}$ a.s. for some nonnegative random variable W_- and we have that*

$$\frac{E e^{-\alpha W_-} - E e^{-\alpha W_+}}{\alpha \eta'(0) \mu p_d} = \frac{\eta(\alpha)}{\alpha \eta'(0)} E e^{-\alpha W_d}. \quad (48)$$

Moreover if any two of EW_d, EW_-, EW_+ are finite, then so is the third and we have that

$$\frac{E e^{-\alpha W_-} - E e^{-\alpha W_+}}{\alpha (EW_+ - EW_-)} = \frac{\eta(\alpha)}{\alpha \eta'(0)} E e^{-\alpha W_d}. \quad (49)$$

The above theorem gives more insight into the distribution and meaning of $W_d + Y_e$, by relating this sum to the random variables W_+ and W_- which successively represent the workload at the ends of down and up periods. For more details regarding the left side of (49), see Theorems 5.1 and 5.2 of [16]. In particular, it is a Laplace-Stieltjes transform of a bona fide distribution if and only if W_- is stochastically smaller than W_+ . This form was also observed and discussed in the M/G/1 queue setting in [18]. Finally, if there are enough assumptions to assure that W_- and $U = W_+ - W_-$ are independent then the left side of (49) becomes

$$Ee^{-\alpha W_-} \frac{1 - Ee^{-\alpha U}}{\alpha EU} . \quad (50)$$

That is, it is the transform of a sum of two independent random variables, the first is W_- and the second has the stationary residual lifetime distribution of U . If we denote this variable by U_e then we have the following decomposition

$$W_- + U_e \sim W_d + Y_e , \quad (51)$$

where we recall that Y_e has the transform $\frac{\eta(\alpha)}{\alpha\eta'(0)}$ and the variables on either side are assumed independent. The special case where this kind of independence (between W_- and U) occurs is discussed in the M/G/1 queue setting in [11]. We also refer the reader to Theorem 4.1 and its proof in [14] for the special case considered there.

We recall that by Theorem 3,

$$W(\infty) \sim I_\ell W_u + (1 - I_\ell)(I(W_u + Y_e) + W_d) . \quad (52)$$

Thus, replacing Y_e by an independent $Y_e^1 \sim Y_e$ and adding Y_e on both sides we have that

$$W(\infty) + Y_e \sim I_\ell(W_u + Y_e) + (1 - I_\ell)(I(W_u + Y_e^1) + (W_d + Y_e)) . \quad (53)$$

With $W_\pm \sim W_d + Y_e$ (a random variable with LST given by the left side of (49)) this implies that

$$W(\infty) + Y_e \sim I_\ell(W_u + Y_e) + (1 - I_\ell)(I(W_u + Y_e^1) + W_\pm) , \quad (54)$$

where the expressions on either side of the equation are independent. Finally, replacing Y_e^1 on the right by Y_e does not change the distribution (due to the indicator I_ℓ) so that

$$W(\infty) + Y_e \sim I_\ell(W_u + Y_e) + (1 - I_\ell)(I(W_u + Y_e) + W_\pm) , \quad (55)$$

where again all variables appearing on the expressions on either side of the equation are assumed independent so that only their marginal

distribution matters. In the special case of $\varphi(\alpha) = \alpha r - \eta(\alpha)$ we replace $I(W_u + Y_e)$ on the right by W_u (see Corollary 1) and obtain

$$\begin{aligned} W(\infty) + Y_e &\sim I_\ell(W_u + Y_e) + (1 - I_\ell)(W_u + W_\pm) \\ &= W_u + I_\ell Y_e + (1 - I_\ell)W_\pm, \end{aligned} \quad (56)$$

and in particular when $\pi_\ell = 0$

$$W(\infty) + Y_e \sim W_u + W_\pm, \quad (57)$$

where, again, throughout all random variables appearing in the expressions on either sides of the equations are assumed independent.

5 Applications to polling systems

In this section we relate our decomposition results to decomposition results for so-called polling systems. A polling system is a single-server multi-queue system, in which the server visits the queues one at a time, typically in a cyclic order. The service discipline at each queue specifies the duration of a visit. E.g., under the exhaustive service discipline, the server visits a queue until it has become empty; under the 1-limited discipline, it serves exactly one customer during a visit. In many applications (e.g., in production systems, where the server is a machine and the customers of a queue are orders of a particular type) it is natural to have nonnegligible switchover times from one queue to the next. Stimulated by a wide variety of applications (not only production systems, but also computer- and communication systems, traffic lights, repair systems), polling models have been extensively studied. It is almost always assumed that the input processes to the queues are independent Poisson processes. For such a situation, it was proven in [2] that the steady state total workload in the polling system *with* switchover times can be decomposed into two independent quantities, viz. (i) the workload in the corresponding polling system *without* switchover times, and (ii) the steady state total amount of work at an epoch the server is not working. Item (i) is the workload in an $M/G/1$ queueing system; the distribution of item (ii) was determined for a few service disciplines in [7]. In [6] the joint steady state workload distribution at arbitrary epochs was expressed in the joint queue length distribution at visit beginning and visit completion epochs. The latter distributions are known for certain polling models, in particular, for polling models in which the service discipline at all queues is of so-called branching type.

The cyclic polling model of [2] was generalized in [4] to the case of a fixed non-cyclic visit order of the queues; again a work decomposition result was derived. A further generalization is contained in

[3]. That paper considers a single-server multi-class system with a work-conserving scheduling discipline as long as the server *is* serving and with a service interruption process (which could correspond to switchover times in a polling system) that does not affect the amount of service time given to a customer or the arrival time of any customer. Furthermore, the arrival process is a batch Poisson process that allows correlations between the numbers of arrivals of the various customer types in a batch. Again a decomposition result was proven: the steady state workload in the model with interruptions is in distribution equal to the sum of two independent quantities, viz. (i) the steady state workload in the corresponding model without interruptions, and (ii) the steady state amount of work at an epoch in which the server is not serving.

Another extension of the cyclic polling model with independent Poisson arrivals was recently studied in [5]. It considers a cyclic polling system with N queues, extending the Poisson arrival process to an N -dimensional Lévy subordinator (so the sample paths are non-decreasing in all coordinates). If a particular queue is being served, then the workload level at that queue behaves as a spectrally positive Lévy process with a negative drift. Another special feature of the model is that the Lévy input process changes at polling and switching instants. A restrictive assumption is that the service discipline at each queue is of branching type. That assumption implies that the N -dimensional workload process at successive instants that the server arrives at the first queue is a *Jirina* process, which is a multi-type continuous-state branching process. The joint steady state workload distribution at such epochs, and subsequently also at arbitrary epochs, is determined in [5]; no workload decomposition is derived. A special case (constant fluid input at all queues) had been studied by Czerniak and Yechiali [8], who also obtained the joint workload distribution at arbitrary epochs. In Section 4 of their paper they point out that, if there is a workload decomposition, the term "(i)" without switchover times is zero because the outflow is larger than the inflow during visit times.

In Section 3 of the present paper, we derive workload transforms and workload decompositions in a system that alternates between up and down periods. The input process is one Lévy process X_u during up periods and another Lévy process X_d during down periods. Our Theorem 2 generalizes exact workload transform results in [6] and [7], where the input process is a sum of independent compound Poisson processes, to the case of a Lévy input process. It complements the exact workload transform result of [5] in the sense that it only gives total workload and does not give a joint transform, but that it does allow more general visit disciplines. Our assumption on the *up* and *down* periods (visit times and interruptions), viz., the assumption that $0 = T_0 \leq S_1 \leq T_1 \leq S_2 \leq T_2 \dots$ is an increasing sequence of

a.s. finite stopping times, in particular includes non-branching service disciplines. Our Corollary 1 generalizes/complements decomposition results for total workload in [2, 3] for polling systems and, more generally, single-server multi-class systems with interruptions. Our Lévy input process generalizes the (batch) Poisson processes of those and other polling papers.

In fact, due to our general setup it seems that under appropriate stability conditions, decomposition results would hold for quite general polling mechanisms. Some examples are cases where the lengths of the switching times depend on the state of the system in various ways (e.g., shorter switching when certain queues are large), or when the decision of when to leave a certain queue may depend on the overall information of the system rather than following a fixed mechanism.

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